Sec. 10.2 Invertibility and Properties of Inverse Functions

Definition of an Inverse Function:

Suppose $Q = f(t)$ is a function with the property that each value of Q determines exactly one value of t. Then f has an **inverse function,** f^{-1}, and

$$f^{-1}(Q) = t \text{ if and only if } Q = f(t).$$

If a function has an **inverse**, it is said to be invertible.

Ex: Find a solution to the equation sin x = 0.8 using an inverse function.

$x = \sin^{-1}(.8)$ or $\sin^{-1}(.8) = x$

$.8 = \sin x$

Ex: Suppose you deposit \$500 into a savings account that pays 4 percent interest compounded annually. The balance, in dollars, in the account after t years is given by $B = f(t) = 500\,(1.04)^t$.

a. Find a formula for $t = f^{-1}(B)$.

$B = 500\,(1.04)^t$

$\frac{B}{500} = 1.04^t$

$\log\left(\frac{B}{500}\right) = \log\left(1.04^t\right)$

$\frac{\log\left(\frac{B}{500}\right)}{\log 1.04} = \frac{t \log(1.04)}{\log(1.04)}$

$t = f^{-1}(B) = \frac{\log\left(\frac{B}{500}\right)}{\log(1.04)}$

$\boxed{t = f^{-1}(B) = \frac{\log B - \log 500}{\log(1.04)}}$

b. What does the inverse function represent in terms of the account?

It will represent the number of years, t, required to reach a particular balance, B.

Ex: Find the inverse of the function $f(x) = 3x/(2x + 1)$.

$y = \frac{3x}{2x+1}$

$x = \frac{3y}{2y+1}$

$x(2y+1) = 3y$

$2xy + x = 3y$

$x = 3y - 2xy$

$x = y(3-2x)$

$\frac{x}{3-2x} = y$

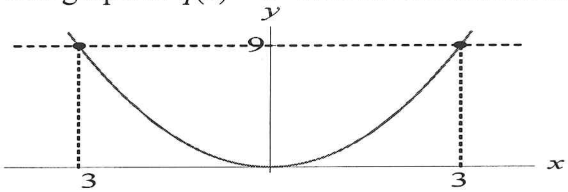
$y = \frac{x}{3-2x}$

$\boxed{f^{-1}(x) = \frac{x}{3-2x}}$

The Horizontal Line Test

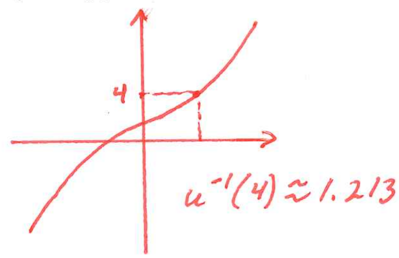
If there is a horizontal line that intersects a function's graph in more than one point, then the function does not have an inverse. If every horizontal line intersects a function's graph at most once, then the function has an inverse.

The graph of $q(x) = x^2$ fails the horizontal line test, so $q(x) = x^2$ has no inverse.



Ex. Let $u(x) = x^3 + x + 1$. Explain why a graph suggests the function is invertible. Assuming u has an inverse, estimate $u^{-1}(4)$.

Passes the horizontal line test.



Example 5

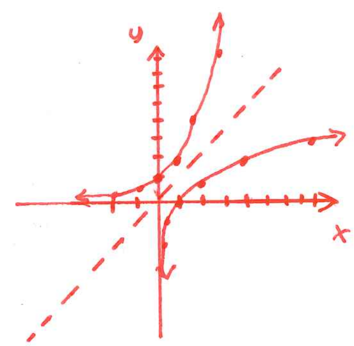
Let $P(x) = 2^x$.

(a) Show that P is invertible.

(b) Find a formula for $P^{-1}(x)$.

(c) Sketch the graphs of P and P^{-1} on the same axes.

(d) What are the domain and range of P and P^{-1}?



$y = 2^x$

$x = 2^y$

$\log x = \log 2^y$

$\log x = y \cdot \log 2$

$\dfrac{\log x}{\log 2} = y$

$P^{-1}(x) = \dfrac{\log x}{\log 2} = \dfrac{1}{\log 2} \cdot \log x$

$\boxed{P^{-1}(x) = 3.222 \log x}$

Domain of P: ALL REALS

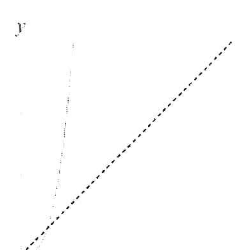
Range of P: $P(x) > 0$

Domain of P^{-1}: $x > 0$

Range of P: ALL REALS

The Graph, Domain, Range and Inverse of a Function:

- Graph of f^{-1} is reflection of graph of f across the line $y = x$.
- Domain of f^{-1} = Range of f
- Range of f^{-1} = Domain of f



If $y = f(x)$ is an invertible function and $y = f^{-1}(x)$ is its inverse, then

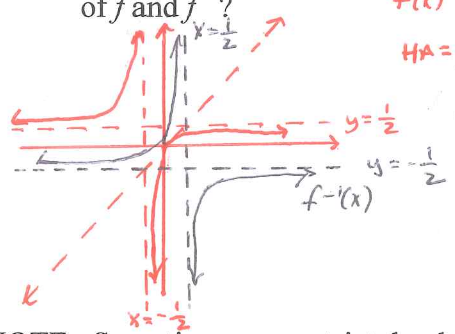
- $f^{-1}(f(x)) = x$ for all values of x for which $f(x)$ is defined,
- $f(f^{-1}(x)) = x$ for all values of x for which $f^{-1}(x)$ is defined.

Ex: (a) Check that $f(x) = x/(2x + 1)$ and $f^{-1}(x) = x/(1 - 2x)$ are inverse functions of each other.

$$f^{-1}(f(x)) = \frac{\frac{x}{2x+1}}{1-2\left(\frac{x}{2x+1}\right)} = \frac{\frac{x}{2x+1}}{1-\frac{2x}{2x+1}} = \frac{\frac{x}{2x+1}}{\frac{2x+1}{2x+1}-\frac{2x}{2x+1}} \to \frac{\frac{x}{2x+1}}{\frac{1}{2x+1}} = \frac{x}{\cancel{2x+1}}\cdot\frac{\cancel{2x+1}}{1} = \frac{x}{1} = x \checkmark$$

$$f(f^{-1}(x)) = \frac{\frac{x}{1-2x}}{2\left(\frac{x}{1-2x}\right)+1} = \frac{\frac{x}{1-2x}}{\frac{2x}{1-2x}+1} = \frac{\frac{x}{1-2x}}{\frac{2x}{1-2x}+\frac{1-2x}{1-2x}} \to \frac{\frac{x}{1-2x}}{\frac{1}{1-2x}} = \frac{x}{1-2x}\cdot\frac{1-2x}{1} = \frac{x}{1} = x \checkmark$$

(b) Graph f and f^{-1} on axes with the same scale. What are the domains and ranges of f and f^{-1}?



$f(x) = \frac{x}{2x+1}$

HA $= y = \frac{1}{2}$

$f^{-1}(x) = \frac{x}{1-2x}$

HA $= y = -\frac{1}{2}$

Domain of $f(x)$: $2x+1 \neq 0$

$2x \neq -1$

$\boxed{\text{ALL REALS EXCEPT } x \neq -\tfrac{1}{2}}$

Range of $f(x)$

$\boxed{\text{ALL REALS EXCEPT } \tfrac{1}{2}}$

Domain of $f^{-1}(x)$:

$1-2x \neq 0$

$-2x \neq -1$

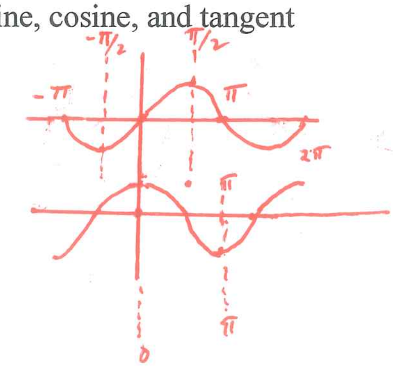
$\boxed{\text{ALL REALS EXCEPT } x \neq \tfrac{1}{2}}$

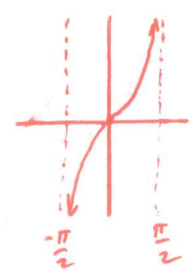
Range of $f^{-1}(x)$:

$\boxed{\text{ALL REALS EXCEPT } -\tfrac{1}{2}}$

{Domain of other function}

NOTE: Sometimes we restrict the domain so that a function will have an inverse. For example, in Section 8.4 we restricted the domains of the sine, cosine, and tangent functions in order to define their inverse functions:





$y = \sin^{-1} x$ if and only if $x = \sin y$ and $-\pi/2 \leq y \leq \pi/2$

$y = \cos^{-1} x$ if and only if $x = \cos y$ and $0 \leq y \leq \pi$

$y = \tan^{-1} x$ if and only if $x = \tan y$ and $-\pi/2 < y < \pi/2$

YOU MAY NEED TO DO THIS TO WORK WITH A SPECIFIC PART OF THE GRAPH!

HW: pg 411-414 #1,6,8,12,18-21,26,33,36,39,41,47